

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2443

THE SIMILARITY LAW FOR HYPERSONIC FLOW ABOUT SLENDER THREE-DIMENSIONAL SHAPES

By Frank M. Hamaker, Stanford E. Neice,
and A. J. Eggers, Jr.

Ames Aeronautical Laboratory
Moffett Field, Calif.

**Reproduced From
Best Available Copy**



Washington
August 1951

DISTRIBUTION STATEMENT A
Approved for Public Release
Distribution Unlimited

20000816 104

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2443

THE SIMILARITY LAW FOR HYPERSONIC FLOW ABOUT
SLENDER THREE-DIMENSIONAL SHAPES

By Frank M. Hamaker, Stanford E. Neice,
and A. J. Eggers, Jr.

SUMMARY

The similarity law for steady, inviscid hypersonic flow about slender three-dimensional shapes is derived in terms of customary aerodynamic parameters. To have similarity of flow, the law states that the lateral dimensions of the shapes in question and their angles with respect to the flight direction must be inversely proportional to their flight Mach numbers. A direct consequence of this law is that the ratio of the local static pressure to the free-stream static pressure is the same at corresponding points in similar flow fields.

The law is applied to the determination of simple expressions for correlating the forces and moments acting on related shapes operating at hypersonic speeds. The shapes considered are wings, bodies, and wing-body combinations. In the special case of inclined bodies of revolution, these expressions are extended to include some significant effects of the viscous cross force.

Results of a limited experimental investigation of the pressures acting on two inclined cones are found to check the law as it applies to bodies of revolution. Further investigation is necessary, however, to determine the range of applicability of the law.

INTRODUCTION

The hypersonic similarity law for steady potential flows about thin airfoil sections and slender nonlifting bodies of revolution was first developed by Tsien in reference 1. The law states that the flows about these shapes will be similar provided (a) the free-stream Mach numbers are large compared to 1, (b) the shapes have the same thickness distributions, and (c) the products of the free-stream Mach numbers and the thickness ratios are the same. Hayes (reference 2) investigated this law from the standpoint of analogous unsteady flows and concluded that it would also apply to nonpotential flows containing pronounced shock waves and vorticity, provided the local Mach number was everywhere

large compared to 1. He also reasoned that similitude could be obtained in hypersonic flows about slender three-dimensional bodies of arbitrary shape; however, the form of the similarity law in terms of customary aerodynamic parameters was not determined.

Ehret, Rossow, and Stevens (reference 3) investigated the hypersonic similarity law for nonlifting bodies of revolution by comparing pressure distributions calculated by means of the method of characteristics. They found the law to be applicable over a wide range of Mach numbers and thickness ratios. Their investigation did not, however, include the effects of vorticity arising from the curvature of the nose shock wave. Rossow (reference 4) continued this investigation and found that the law was equally valid when the effects of vorticity were included in the calculations. These findings corroborated, in part, the observations of Hayes and indicated that the law may be used with confidence to investigate the aerodynamic characteristics of nonlifting bodies of revolution at hypersonic speeds.

With the successful application of the hypersonic similarity law to nonlifting bodies of revolution, it appeared desirable to determine the form of the law, in terms of customary aerodynamic parameters, for slender three-dimensional bodies of arbitrary shape. An investigation of the more general law was therefore undertaken. The purpose of this paper is to present the results of this study.

SYMBOLS

a	speed of sound
A	characteristic reference area of body ($A = b t$)
b	characteristic span or width of body
c	characteristic chord or length of body
c_{d_c}	section drag coefficient of circular cylinder with axis perpendicular to the flow
\hat{c}_{d_c}	mean c_{d_c} for a body of revolution
c_C	side-force coefficient $\left(\frac{\text{side force}}{\frac{1}{2} \rho_0 V_0^2 A} \right)$
\tilde{c}_C	side-force parameter
c_D	drag coefficient $\left(\frac{\text{drag}}{\frac{1}{2} \rho_0 V_0^2 A} \right)$

- \tilde{C}_D drag parameter
- C_l rolling-moment coefficient $\left(\frac{\text{rolling moment}}{\frac{1}{2} \rho_0 V_0^2 A_b} \right)$
- \tilde{C}_l rolling-moment parameter
- C_L lift coefficient $\left(\frac{\text{lift}}{\frac{1}{2} \rho_0 V_0^2 A} \right)$
- \tilde{C}_L lift parameter
- C_m pitching-moment coefficient $\left(\frac{\text{pitching moment}}{\frac{1}{2} \rho_0 V_0^2 A_c} \right)$
- \tilde{C}_m pitching-moment parameter
- C_n yawing-moment coefficient $\left(\frac{\text{yawing moment}}{\frac{1}{2} \rho_0 V_0^2 A_b} \right)$
- \tilde{C}_n yawing-moment parameter
- f dimensionless perturbation potential function
- F viscous force or moment function
- g dimensionless body shape function
- G body shape function
- $\bar{i}, \bar{j}, \bar{k}$ unit vectors along coordinate axes x, y, z , respectively
- l, m, n direction cosines of the vector \bar{N} with respect to the x, y, z axes, respectively
- K_b
 K_t
 K_α
 K_β
 K_δ
- Mach number
- \bar{N} unit outer normal to surface of body
- p static pressure
- r radius of body of revolution at any station x

R_c cross Reynolds number based on maximum body diameter and the component of the free-stream velocity normal to the body axis
 s cross force per unit length
 t characteristic thickness or depth of body
 u, v, w components of velocity, V , in the direction of the x, y, z axes, respectively
 v resultant velocity
 x, y, z Cartesian coordinates
 α angle of attack
 β angle of sideslip
 γ ratio of specific heats
 δ angle of roll
 ξ, η, ζ dimensionless coordinates corresponding to x, y, z , respectively
 θ orifice location on the test cones
 ρ stream density
 Φ perturbation velocity potential

Subscripts

v refers to viscous cross-force effects
 o refers to free-stream conditions
 $1, 2, 3$ refers to different functions F , \tilde{C}_m , or \tilde{C}_n , except as noted

Superscript

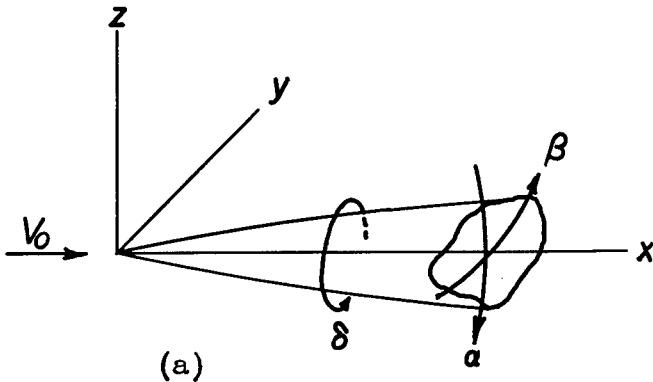
- refers to vector quantities

DEVELOPMENT OF THE SIMILARITY LAW
FOR INVISCID THREE-DIMENSIONAL FLOW

The following assumptions are made in this analysis: (1) the Mach number of the uniform free stream is large compared to 1 (i.e., the flow is hypersonic), (2) the disturbance velocities are small compared to the free-stream velocity, and (3) the flow is of the steady potential type. It is clear from the first two assumptions that the analysis is strictly applicable only to slender shapes in hypersonic flow. As was pointed out in the introduction, however, the last assumption should not restrict the range of applicability of the results to potential flows. The purpose of making this assumption is to simplify the analysis.

A slender body is oriented in x, y, z space as shown in sketch (a) with the free-stream velocity V_0 directed along the x axis.

The general differential equation of motion for steady flow about the body can be written in the following form:



$$(a^2 - u^2)u_x + (a^2 - v^2)v_y + (a^2 - w^2)w_z - uv(u_y + v_x) - \\ vw(v_z + w_y) - wu(w_x + u_z) = 0 \quad (1)$$

where the condition of irrotationality requires that

$$u_y = v_x, \quad v_z = w_y, \quad w_x = u_z \quad (2)$$

As a consequence of equation (2), a perturbation velocity potential, Φ , can be defined as follows:

$$u = V_0 + \Phi_x, \quad v = \Phi_y, \quad w = \Phi_z \quad (3)$$

The energy equation, relating free-stream and local conditions on the body, can be written in the following form:

$$a_0^2 + \frac{\gamma-1}{2} V_0^2 = a^2 + \frac{\gamma-1}{2} (u^2 + v^2 + w^2) \quad (4)$$

Introducing the perturbation potential expressions of equation (3), equation (4) then becomes

$$a_o^2 + \frac{\gamma-1}{2} V_o^2 = a^2 + \frac{\gamma-1}{2} (V_o^2 + 2V_o\Phi_x + \Phi_x^2 + \Phi_y^2 + \Phi_z^2) \quad (5)$$

If equations (2), (3), and (5) are now introduced into equation (1), the steady-state, three-dimensional potential equation of motion is obtained as follows:

$$\begin{aligned} & \left[a_o^2 - \frac{\gamma-1}{2} (2V_o\Phi_x + \Phi_x^2 + \Phi_y^2 + \Phi_z^2) - V_o^2 - 2V_o\Phi_x - \Phi_x^2 \right] \Phi_{xx} + \\ & \left[a_o^2 - \frac{\gamma-1}{2} (2V_o\Phi_x + \Phi_x^2 + \Phi_y^2 + \Phi_z^2) - \Phi_y^2 \right] \Phi_{yy} + \\ & \left[a_o^2 - \frac{\gamma-1}{2} (2V_o\Phi_x + \Phi_x^2 + \Phi_y^2 + \Phi_z^2) - \Phi_z^2 \right] \Phi_{zz} - \\ & 2(V_o\Phi_y + \Phi_x\Phi_y)\Phi_{xy} - 2(V_o\Phi_z + \Phi_x\Phi_z)\Phi_{xz} - 2\Phi_y\Phi_z\Phi_{yz} = 0 \end{aligned} \quad (6)$$

For hypersonic flow about slender shapes, Φ_x , Φ_y , Φ_z , and a_o are small compared to V_o , and a simple analysis further indicates that Φ_x is small compared to Φ_y and Φ_z .¹ Accordingly, the exact potential equation is simplified by neglecting, in general, all terms of higher order than Φ_y^2 and Φ_z^2 , and by neglecting, in particular, all terms except $-V_o^2$ in the coefficient of Φ_{xx} . Equation (6) may therefore be reduced to the form

$$\begin{aligned} & M_o^2 \Phi_{xx} - \left[1 - (\gamma-1) \frac{M_o}{a_o} \Phi_x - \frac{\gamma+1}{2a_o^2} \Phi_y^2 - \frac{\gamma-1}{2a_o^2} \Phi_z^2 \right] \Phi_{yy} - \\ & \left[1 - (\gamma-1) \frac{M_o}{a_o} \Phi_x - \frac{\gamma-1}{2a_o^2} \Phi_y^2 - \frac{\gamma+1}{2a_o^2} \Phi_z^2 \right] \Phi_{zz} + \\ & 2 \frac{M_o}{a_o} \Phi_y \Phi_{xy} + 2 \frac{M_o}{a_o} \Phi_z \Phi_{xz} + \frac{2}{a_o^2} \Phi_y \Phi_z \Phi_{yz} = 0 \end{aligned} \quad (7)$$

¹ To illustrate, for two-dimensional flows of the type considered in this analysis the compatibility equations, which hold along characteristic lines, take on the form $\Delta V = \pm \frac{1}{M} \Delta v$. Since M is large compared

to 1, this equation shows that ΔV is small compared to Δv . It follows, then, that Φ_x is small compared to Φ_z . From a different point of view, the statement that Φ_x is small compared to Φ_z is just another way of stating a well-known property of hypersonic flows, namely, that although the direction of the resultant velocity vector may change appreciably, the magnitude of the velocity vector changes only slightly.

This relation is employed as the equation of motion in the following analysis. The boundary conditions remain to be determined.

The shape of a slender three-dimensional body is defined in its reference position² in the flow field by the functional relation

$$G(x, y, z) = 0 \quad (8)$$

The unit normal at a point on the surface is given by the vector

$$\bar{N} = l\bar{i} + m\bar{j} + n\bar{k} \quad (9)$$

and the requirement that the body be slender is satisfied by the restriction

$$l \ll 1 \quad (10)$$

at all points. One boundary condition is, of course, the requirement that there be no normal component of flow at the surface of the body. This condition is satisfied for the body in its reference position if the relation

$$\bar{V} \cdot \bar{N} = (V_0 + \Phi_x)l + \Phi_y m + \Phi_z n = 0$$

and hence

$$(V_0 + \Phi_x)G_x + \Phi_y G_y + \Phi_z G_z = 0 \quad (11)$$

holds everywhere on the surface. This expression can readily be generalized to include steady motion at small angles of attack, sideslip, and roll. Rotating the body to these angles relative to the wind introduces a corresponding rotation of the normals to the body.³ In terms of the direction cosines of the original normals, then, the expression for the rotated normals may be given in the form

$$\bar{N}' = (l + m\beta - n\alpha)\bar{i} + (m + n\delta - l\beta)\bar{j} + (n + l\alpha - m\delta)\bar{k} \quad (12)$$

² The body is defined as being in its reference position when the nose coincides with the origin of the coordinate system and the angles of attack, sideslip, and roll are zero.

³ In general, such rotations are not commutative; that is, the result is not the same when the sequence of the rotations is changed. This difficulty is avoided by restricting the analysis to consider only terms of the first order in the angles of rotation. This restriction is consistent with the initial assumptions but does not have to be made in the case of the angle of roll as will be discussed in greater detail later in the report.

Imposing the requirement specified by equation (10), equation (12) is further reduced to the form

$$\bar{N}' = (l + m\beta - n\alpha)\bar{i} + (m + n\delta)\bar{j} + (n - m\delta)\bar{k} \quad (13)$$

If the vector \bar{N} in equation (11) is replaced by \bar{N}' as defined in this expression, then the desired generalized boundary condition on the surface of the body, is given by the equation

$$V_0(G_x + \beta G_y + \alpha G_z) + \Phi_y(G_y + \delta G_z) + \Phi_z(G_z - \delta G_y) = 0 \quad (14)$$

In this equation the derivatives of G are, of course, evaluated on the surface of the body in the reference position, while the derivatives of Φ are evaluated at corresponding points on the body in its rotated position. The remaining boundary condition is, of course,

$$\Phi_x = \Phi_y = \Phi_z = 0 \quad \text{at } x = -\infty \quad (15)$$

In order to obtain the similarity law for flow about related bodies, it is convenient to express the equations of motion and boundary conditions in a nondimensional form. A dimensionless coordinate system is therefore introduced with the affine transformation

$$\xi = \frac{x}{c}, \quad \eta = \frac{y}{b}, \quad \zeta = \frac{z}{t} \quad (16)$$

and a nondimensional perturbation potential function is defined by the relation⁴

$$f(\xi, \eta, \zeta) = \frac{\Phi(x, y, z)}{a_0 M_0 c \left(\frac{t}{c} \right)^2} \quad (17)$$

where c , b , and t are a characteristic length, width and height of a body, respectively. Under the coordinate transformation given above, equation (8) takes the form

$$g(\xi, \eta, \zeta) = 0 \quad (18)$$

Substituting equations (16) and (17) into equation (7) there is then obtained for the equation of motion

$$K_t^2 f_{\xi\xi} - \left[1 - (\gamma-1) K_t^2 f_\xi - \frac{\gamma+1}{2} \left(\frac{K_t}{K_b} \right)^2 K_t^2 f_\eta^2 - \frac{\gamma-1}{2} K_t^2 f_\xi^2 \right] \left(\frac{K_t}{K_b} \right)^2 f_{\eta\eta} -$$

⁴The function chosen here differs from those employed by Tsien and Hayes. It has the advantage of simplifying the expressions for the boundary conditions.

$$\left[1 - (\gamma-1)K_t^2 f_\xi - \frac{\gamma-1}{2} \left(\frac{K_t}{K_b} \right)^2 K_t^2 f_\eta^2 - \frac{\gamma+1}{2} K_t^2 f_\zeta^2 \right] f_{\xi\xi} + \\ 2 \left(\frac{K_t}{K_b} \right)^2 K_t^2 f_\eta f_\xi \eta + 2 K_t^2 f_\zeta f_\xi \zeta + \left(\frac{K_t}{K_b} \right)^2 K_t^2 f_\eta f_\zeta f_{\eta\zeta} = 0 \quad (19)$$

In an analogous manner, equations (14) and (15) for the boundary conditions assume the nondimensional forms

$$g_\xi + g_\eta \frac{K_\beta}{K_b} - g_\zeta \frac{K_\alpha}{K_t} + \left(\frac{K_t}{K_b} \right)^2 f_\eta \left(g_\eta + g_\zeta K_\delta \frac{K_b}{K_t} \right) + \\ f_\zeta \left(g_\zeta - g_\eta K_\delta \frac{K_t}{K_b} \right) = 0 \quad (20)$$

on the surface, and

$$f_\xi = f_\eta = f_\zeta = 0 \quad \text{at} \quad \xi = -\infty \quad (21)$$

where the hypersonic similarity parameters for a constant value of γ are given as follows:

$$K_t = M_\infty \frac{t}{c} \quad (22)$$

$$K_b = M_\infty \frac{b}{c} \quad (23)$$

$$K_\alpha = M_\infty \alpha \quad (24)$$

$$K_\beta = M_\infty \beta \quad (25)$$

$$K_\delta = \delta \quad (26)$$

With equations (19) through (26), the similarity law for inviscid hypersonic flow about slender shapes can be deduced, for it is clear that the flow now depends only on the dimensionless shape function g (i.e., the thickness distribution of a shape or body in the flow) and the similarity parameters previously given. Thus the law may be stated as follows: For bodies described by the same dimensionless function and immersed in flows such that the same values of the similarity parameters are obtained, the disturbance flow fields are defined by the same dimensionless perturbation potential function, and are, therefore, similar. Thus for similarity of flow about bodies, it is only necessary

that their lateral dimensions and angles with respect to the flow direction be inversely proportional to the Mach number of the flow.

This statement of the law is essentially a generalization of that originally presented by Tsien. The new similarity parameters K_b , K_α , K_β , and K_δ define additional restrictions on the shapes and attitudes of related bodies;⁵ however, the similarity parameter K_t (and the restriction imposed by it) is the same as the one in reference 1, obtained from the considerations of two-dimensional and axially symmetric flows. In regard to the new similarity parameters, attention is called to K_δ which, it is noticed, does not contain M_∞ . The roll angle is the same, then, for related bodies in similar hypersonic flows. This result could have been deduced intuitively, and it seems equally clear that if the rotations to angles of attack, sideslip, and roll are required to be in the same sequence (see footnote 3), then the result is also valid for arbitrarily large angles of roll.

APPLICATIONS OF THE SIMILARITY LAW AND DISCUSSION OF RESULTS

In the preceding section the hypersonic similarity law was developed in a general form. The law is employed in this section to correlate the physical properties of similar flow fields and the aerodynamic characteristics of some related shapes of practical interest.

Some effects of viscosity are considered in the investigation of the aerodynamic characteristics for inclined bodies of revolution. The assumption of inviscid flow is, however, retained elsewhere in this study.

Correlation of the Physical Properties of Similar Flow Fields

In aerodynamic studies, perhaps the most important physical property of a fluid is the static pressure. This pressure at any point in a flow field of the type under consideration is given by the relation

$$p = p_\infty \left(\frac{1 + \frac{\gamma-1}{2a_0^2} V_\infty^2}{1 + \frac{\gamma-1}{2a^2} V^2} \right)^{\frac{\gamma}{\gamma-1}}$$

⁵The terms "related bodies" will be used, henceforth, to identify bodies that are described by the same shape function g .

Simplifying this equation to include only terms of the proper order and transforming the resulting expression to nondimensional form yields the following relation:

$$\frac{p}{p_0} = \left\{ 1 - (\gamma-1)K_t^2 f_\xi - \frac{\gamma-1}{2} \left[K_t^2 \left(\frac{K_t}{K_b} \right)^2 f_\eta^2 + K_t^2 f_\zeta^2 \right] \right\}^{\frac{\gamma}{\gamma-1}}$$

The derivatives of f are, however, functions only of the similarity parameters and the dimensionless coordinates; therefore, this expression may be written as

$$\frac{p}{p_0} = \frac{p}{p_0} (\xi, \eta, \zeta; K_t, K_b, K_a, K_\beta, K_\delta) \quad (27)$$

It is clear from this relation that for similar flows, the ratio of the local to the free-stream static pressure is the same at corresponding points (ξ, η, ζ) in the flow fields. A direct consequence of this rule is that the center of pressure is at the same (ξ, η, ζ) location on related bodies in similar hypersonic flows. It may easily be shown that this rule can also be applied to relate other physical properties of similar flow fields, such as temperatures, densities and Mach numbers.

Correlation of the Aerodynamic Characteristics of Some Related Shapes

Bodies of Revolution.- For bodies of revolution, equation (27) reduces to the form⁶

$$\frac{p}{p_0} = \frac{p}{p_0} (\xi, \eta, \zeta; K_t, K_a)$$

where K_b is eliminated as it is identical to K_t . This equation is integrated in the usual manner to obtain the lift, drag, and pitching-moment coefficients of related bodies. It is convenient to write the expressions for these coefficients in the following forms:

⁶Because of the axial symmetry of bodies of revolution only angles of attack are considered. This consideration obviates a discussion of force and moment characteristics at angles of sideslip or combined angles of attack and sideslip, while roll, of course, has no meaning. It is clear, then, that the similarity parameters K_β and K_δ are eliminated from this analysis.

$$\left. \begin{aligned} M_0 C_L &= \tilde{C}_L = \tilde{C}_L(K_t, K_\alpha) \\ M_0^2 C_D &= \tilde{C}_D = \tilde{C}_D(K_t, K_\alpha) \\ M_0 C_m &= \tilde{C}_m = \tilde{C}_m(K_t, K_\alpha) \end{aligned} \right\} (28)$$

Where \tilde{C}_L , \tilde{C}_D , and \tilde{C}_m are designated lift, drag, and pitching-moment parameters, respectively.⁷ It is apparent from these relations that the corresponding force and moment parameters have identical values for related bodies of revolution provided the corresponding similarity parameters have identical values. It will now be shown that this conclusion can be generalized to include the significant effects of the viscous cross forces on related inclined bodies.

The viscous cross force arises from the flow (usually partially separated) of the boundary-layer transverse to the body axis. A method of estimating this force along with the lift, drag, and pitching-moment coefficients associated with it has been suggested by Allen in reference 5, and is presented in the appendix of the present paper. The resulting expressions for these coefficients (see equation (c) in the appendix) are transformed to the nondimensional form and the following relations are obtained:

$$\left. \begin{aligned} M_0 C_{L_V} &= \hat{c}_{d_C} F_1(K_t, K_\alpha) \\ M_0^2 C_{D_V} &= \hat{c}_{d_C} F_2(K_t, K_\alpha) \\ M_0 C_{M_V} &= \hat{c}_{d_C} F_3(K_t, K_\alpha) \end{aligned} \right\} (29)$$

For slender bodies of revolution of the type under consideration, \hat{c}_{d_C} is primarily a function of the Mach number and Reynolds number of the flow component normal to the body axis. Consequently, these expressions can be reduced to the form

$$\left. \begin{aligned} M_0 C_{L_V} &= \tilde{C}_{L_V} = \tilde{C}_{L_V}(K_t, K_\alpha, R_c) \\ M_0^2 C_{D_V} &= \tilde{C}_{D_V} = \tilde{C}_{D_V}(K_t, K_\alpha, R_c) \\ M_0 C_{m_V} &= \tilde{C}_{m_V} = \tilde{C}_{m_V}(K_t, K_\alpha, R_c) \end{aligned} \right\} (30)$$

where R_c is the cross Reynolds number. It is clear when comparing these relations with those of equation (28) that the conclusion drawn from the latter relations applies with equal validity when viscous cross-flow effects are considered, provided that R_c is included as a similarity parameter.⁸

⁷If the angle of attack is zero, $K_\alpha = 0$ and the expression for the drag parameter reduces to a form equivalent to that obtained by Tsien.

⁸It is assumed that the viscous flow considered here does not significantly influence the potential, inviscid flow discussed previously. Hence the force and moment coefficients resulting from these flows may be superimposed.

A limited experimental check of the similarity law for bodies of revolution has been made in the Ames 10- by 14-inch supersonic wind tunnel. Two cones having thickness ratios of 0.333 and 0.204 were tested at Mach numbers of 2.75 and 4.46, respectively; thus the value of K_t was 0.91. Equipment for measuring forces and moments was not available at the time of these tests; therefore, pressures only were measured on the cones. These measurements were made at the locations shown in figure 1 for angles of attack up to 5° . Overlapping values of K_α up to 14° were thus obtained. The ranges of cross-flow Reynolds numbers covered in the tests are shown in figure 2, and it is evident that identical values of R_c could not be obtained for the two cones at the same values of K_α .

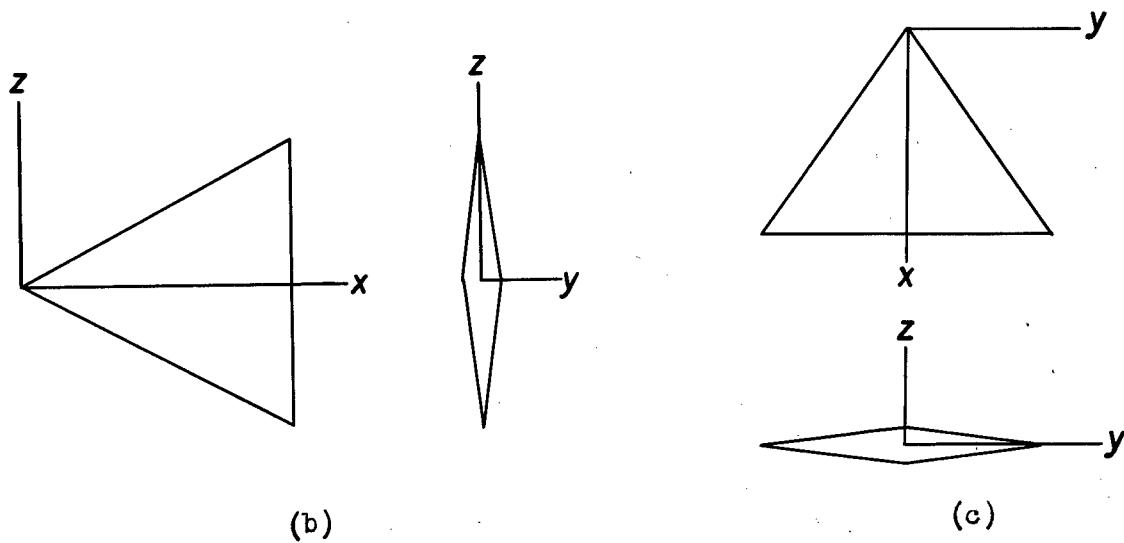
Experimentally determined pressure ratios are shown in figure 3 as a function of K_α . Agreement with the prediction of the similarity law is generally observed, in that the values of p/p_0 for corresponding points on the two bodies lie essentially along the same curve. The exception to this agreement is on the lee sides of the cones ($\theta=180^\circ$) where it is noted that significantly different curves are defined. This difference is believed to be the result of dissimilar flow separation from the two cones, caused in turn by the marked differences in the cross-flow Reynolds number previously mentioned. Separation phenomena should be essentially similar at identical cross-flow Reynolds numbers, in which case the corresponding values of p/p_0 should agree.

Wings, Bodies, and Wing-Body Combinations. - The general form of the similarity law must be employed in this phase of the investigation. In order, then, to obtain expressions for the force and moment parameters of wings, bodies, and wing-body combinations, it is necessary to integrate equation (27) over related, but otherwise arbitrary shapes. The resulting expressions are

$$\left. \begin{aligned} M_0 C_L &= \tilde{C}_L = \tilde{C}_L(K_t, K_b, K_\alpha, K_\beta, K_\delta) \\ M_0^2 C_D &= \tilde{C}_D = \tilde{C}_D(K_t, K_b, K_\alpha, K_\beta, K_\delta) \\ M_0 C_C &= \tilde{C}_C = \tilde{C}_C(K_t, K_b, K_\alpha, K_\beta, K_\delta) \\ M_0 C_m &= \tilde{C}_m = \tilde{C}_{m_1}(K_t, K_b, K_\alpha, K_\beta, K_\delta) + \frac{1}{M_0^2} \tilde{C}_{m_2}(K_t, K_b, K_\alpha, K_\beta, K_\delta) \\ C_n &= \tilde{C}_n = \tilde{C}_{n_1}(K_t, K_b, K_\alpha, K_\beta, K_\delta) + \frac{1}{M_0^2} \tilde{C}_{n_2}(K_t, K_b, K_\alpha, K_\beta, K_\delta) \\ M_0 C_l &= \tilde{C}_l = \tilde{C}_l(K_t, K_b, K_\alpha, K_\beta, K_\delta) \end{aligned} \right\} \quad (31)$$

It is clear from the equations for the pitching-moment and yawing-moment parameters that these two parameters cannot be correlated for related wings, bodies or wing-body combinations of slender, but otherwise

completely arbitrary shape.⁹ Correlation can be achieved, however, if two restrictions are placed on the shapes of these configurations. For the case of pitching moment, the restriction is that the l direction cosines of the outer normals to the surface must, in general, be small compared to the corresponding n direction cosines. Thus, for example, vertical fins (alone) having surface slopes in the chordwise direction generally of the same order of magnitude as the slopes in the depthwise direction are eliminated from consideration. Such a shape is shown in sketch (b). In the case of yawing moment, the restriction



is that l must, in general, be small compared to m . Thus, for example, wings, as shown in sketch (c), having chordwise slopes generally of the same order of magnitude as the spanwise slopes, are eliminated from consideration. With these restrictions, the terms in the relations for $M_{O,Cm}$ and C_n containing $1/M_0^2$ as their coefficient may be neglected, and thus correlating expressions for these parameters are obtained. In this case the more general consequence of the similarity law for inviscid hypersonic flow is apparent; namely, the corresponding force and moment parameters have identical values for related wings, bodies or wing-body combinations, provided the corresponding similarity parameters have identical values.¹⁰

It is of interest to examine these relations as they apply to thin wings. If, for spanwise symmetric wings, only angle of attack is

⁹The second term on the right in the relations for $M_{O,Cm}$ and C_n arises from the moment due to the nonsymmetry of the drag force.

¹⁰It is clear that the conclusion drawn from equation (28), applying to bodies of revolution, is a restricted form of this statement. It is not evident, however, that this statement can be generalized to include significant viscous effects, as was possible with the aforementioned conclusion.

considered, the similarity parameters K_β and K_δ vanish and only three of the aerodynamic coefficients remain. The corresponding force and moment parameters are reduced to the forms¹¹

$$\left. \begin{aligned} M_0 C_L &= \tilde{C}_L = \tilde{C}_L(K_t, K_b, K_\alpha) \\ M_0^2 C_D &= \tilde{C}_D = \tilde{C}_D(K_t, K_b, K_\alpha) \\ M_0 C_m &= \tilde{C}_m = \tilde{C}_m(K_t, K_b, K_\alpha) \end{aligned} \right\} \quad (32)$$

These relations also apply, of course, to wing sections. In this case, b and therefore K_b are infinite and it is seen from equations (19) and (20), that the terms involving K_b vanish yielding the two-dimensional equations for hypersonic flow. The similarity parameter K_b is thus eliminated from equation (32). This result is equivalent to that presented in reference 1.¹²

Of practical importance is the conclusion to be drawn from the dimensionless equation of motion as it applies to thin wings. It is noticed in the equation that the parameter K_b always appears in the form $\left(\frac{K_t}{K_b}\right)^2 = \frac{t^2}{b^2}$. If b is of the same order of magnitude as c , then, consistent with the other approximations made in developing this equation, the terms involving $\left(\frac{K_t}{K_b}\right)^2$ are to be neglected. Performing this operation, however, yields the equation of motion for two-dimensional flow. Thus it is indicated that, if the aspect ratio is of the order of magnitude of one or greater, hypersonic flow about wings may be treated approximately as a two-dimensional-flow problem. The latter problem is, of course, relatively simple to solve.

A particular example is chosen to illustrate the application of the similarity law to wing-body combinations, which may be thought of, for this purpose, simply as irregular shapes. In figure 4 are shown two related cruciform wing and body combinations at related angles of attack. It is seen that in going from a Mach number of 4 to a Mach number of 8, the wing and body thickness, the wing spans, and the angle of attack are decreased by one-half in order to maintain similarity of flow. The effects of the changes on some of the aerodynamic coefficients are also shown in the figure.

¹¹Parameters equivalent to these were obtained by Tsien and, although not published, were presented in the form of lecture notes. As so often happens, these notes were brought to the attention of the authors after completion of this investigation.

¹²The exponents of M_0 obtained here are different from those obtained in reference 1, because $b t$ is used as a reference area, rather than $c b$.

CONCLUDING REMARKS

The similarity law for steady, inviscid hypersonic flow about slender three-dimensional shapes has been derived in terms of customary aerodynamic parameters. To have similarity of flow, the law states that the lateral dimensions of the shapes in question and their angles with respect to the flight direction must be inversely proportional to their flight Mach numbers. A direct consequence of this law is that the ratio of the local static pressure to the free-stream static pressure is the same at corresponding points in similar flow fields. With the aid of this law, simple expressions were obtained for correlating the forces and moments acting on related shapes in hypersonic flows. The shapes treated were wings, bodies, and wing-body combinations. In the case of inclined bodies of revolution, these expressions were generalized to include the significant effects of the viscous cross force. The law, as it applies to bodies of revolution, was subjected to a limited experimental check by comparing pressures measured on two inclined cones in related flows. Theory and experiment were in good agreement except on the lee sides of the cones where the dissimilar cross-flow Reynolds numbers would be expected to yield dissimilar separated flows.

The range of applicability of the law for practical three-dimensional shapes appears to merit investigation. If this range is relatively as wide as the corresponding range for noninclined bodies of revolution, the law should prove of value in correlating experimental data, and in simplifying theoretical calculations of the aerodynamic characteristics for families of these shapes.

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif., June 5, 1951.

APPENDIX

FORCES AND MOMENTS DUE TO VISCOUS CROSS FLOWS
ON BODIES OF REVOLUTION

In reference 6, Prandtl demonstrated that laminar viscous flows over infinitely long inclined cylinders may be treated by considering, independently, the components of the flow normal and parallel to the axis of the cylinder. Jones, in reference 7, applied this concept to the study of boundary-layer flows over yawed cylinders. The work of Prandtl and Jones suggests, as indicated by Allen in reference 5, that the cross force on slender inclined bodies of revolution may be estimated in the following manner: Each cross section of the body is treated as an element of an infinite cylinder of the same radius. The cross force per unit length on such a cylinder is given by the following equation:

$$s_v = r c_{d_c} \rho_0 V_0^2 \sin^2 \alpha \quad (A1)$$

The incremental lift, drag, and moment produced by this cross force are then given by the relations

$$\left. \begin{aligned} \text{lift} &= r c_{d_c} \rho_0 V_0^2 \sin^2 \alpha \cos \alpha \\ \text{drag} &= r c_{d_c} \rho_0 V^2 \sin^2 \alpha \\ \text{moment} &= r x c_{d_c} \rho_0 V_0^2 \sin^2 \alpha \end{aligned} \right\} \quad (A2)$$

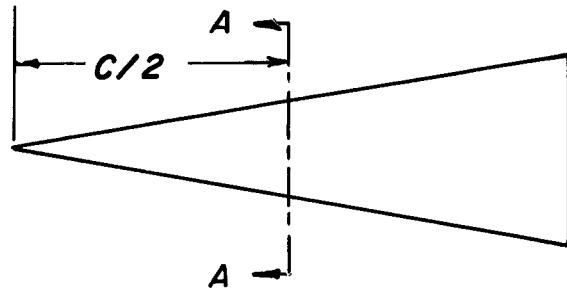
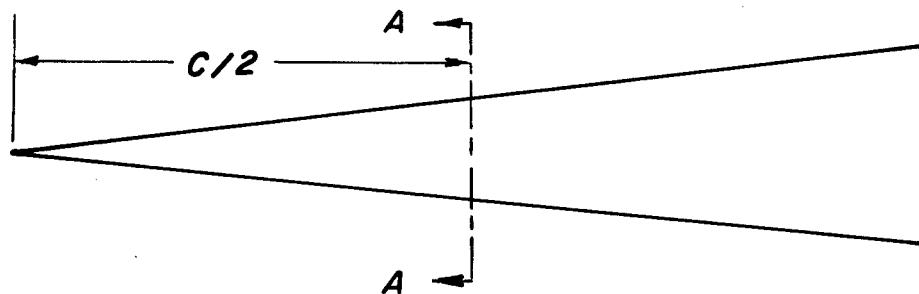
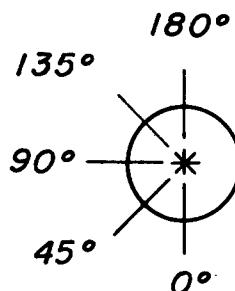
Retaining leading terms in α and integrating over the body, where $r = r(x)$, the aerodynamic coefficients are given by the equations

$$\left. \begin{aligned} C_{L_V} &= \frac{2\hat{c}_{d_c} \alpha^2}{A} \int_0^c r dx \\ C_{D_V} &= \frac{2\hat{c}_{d_c} \alpha^3}{A} \int_0^c r dx \\ C_{m_V} &= \frac{2\hat{c}_{d_c} \alpha^2}{Ac} \int_0^c rx dx \end{aligned} \right\} \quad (A3)$$

where the reference area is proportional to the maximum cross-sectional area of the body and the reference length is the body length. The coefficient \hat{c}_{d_c} is the mean c_{d_c} for the body of revolution, and has therefore been taken outside the integral.

REFERENCES

1. Tsien, Hsue-shen: Similarity Laws of Hypersonic Flows, Journal of Mathematics and Physics, vol. 25, no. 3, Oct. 1946, pp. 252-259.
2. Hayes, Wallace D.: On Hypersonic Similitude. Quarterly of Applied Mathematics, vol. V, no. 1, April 1947, pp. 105-106.
3. Ehret, Dorris M., Rossow, Vernon J., and Stevens, Victor I.: An Analysis of the Applicability of the Hypersonic Similarity Law to the Study of Flow about Bodies of Revolution at Zero Angle of Attack. NACA TN 2250, 1950.
4. Rossow, Vernon J.: Applicability of the Hypersonic Similarity Rule to Pressure Distributions Which Include the Effects of Rotation for Bodies of Revolution at Zero Angle of Attack. NACA TN 2399, 1951.
5. Allen, H. Julian: Pressure Distribution and Some Effects of Viscosity on Slender Inclined Bodies of Revolution. NACA TN 2044, 1950.
6. Prandtl, L.: On Boundary Layers in Three-Dimensional Flow. Ministry of Aircraft Production, Volkenrode (Reports and Translations No. 64) May 1, 1946.
7. Jones, Robert T.: Effects of Sweepback on Boundary Layer and Separation. NACA Rep. 884, 1947. (Formerly NACA TN 1402.)

(a) $t/c = .333$ (b) $t/c = .204$ (c) Orifice location, θ
in transverse plane, A-AFigure 1.— Location of orifices on two cones tested at $K_f = 0.91$.

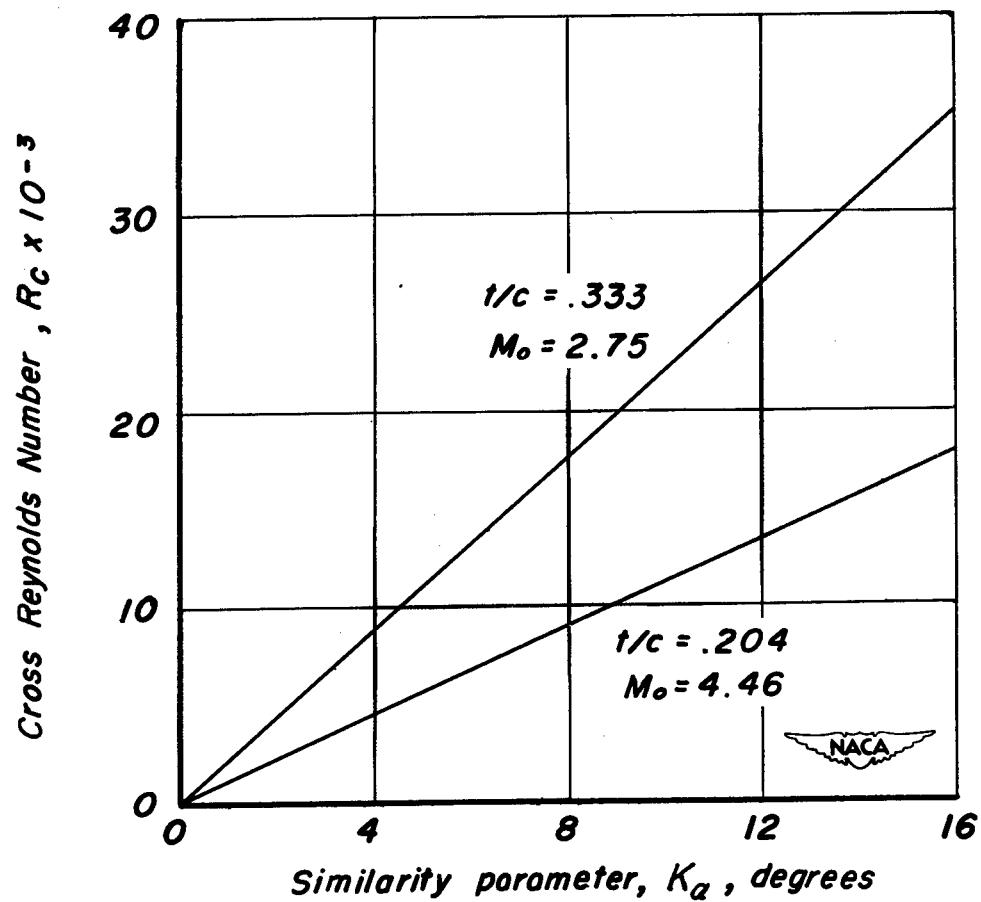


Figure 2.— Variation of cross Reynolds number, R_c , with K_α for two cones tested at $K_f = 0.91$.

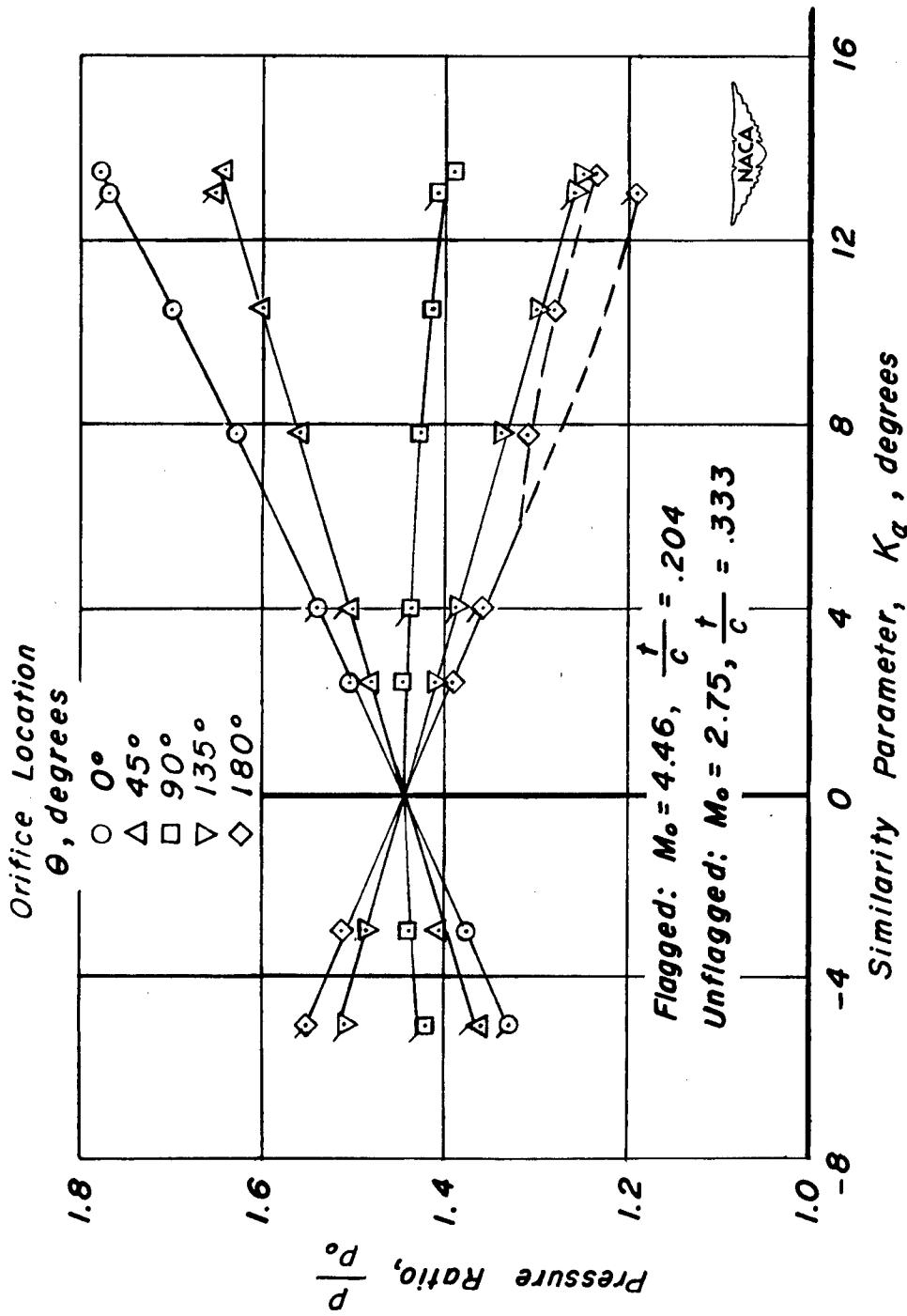


Figure 3—Variation of pressure ratio, $\frac{P_a}{P_0}$, with κ_α for two cones tested at $K_f = 0.91$.

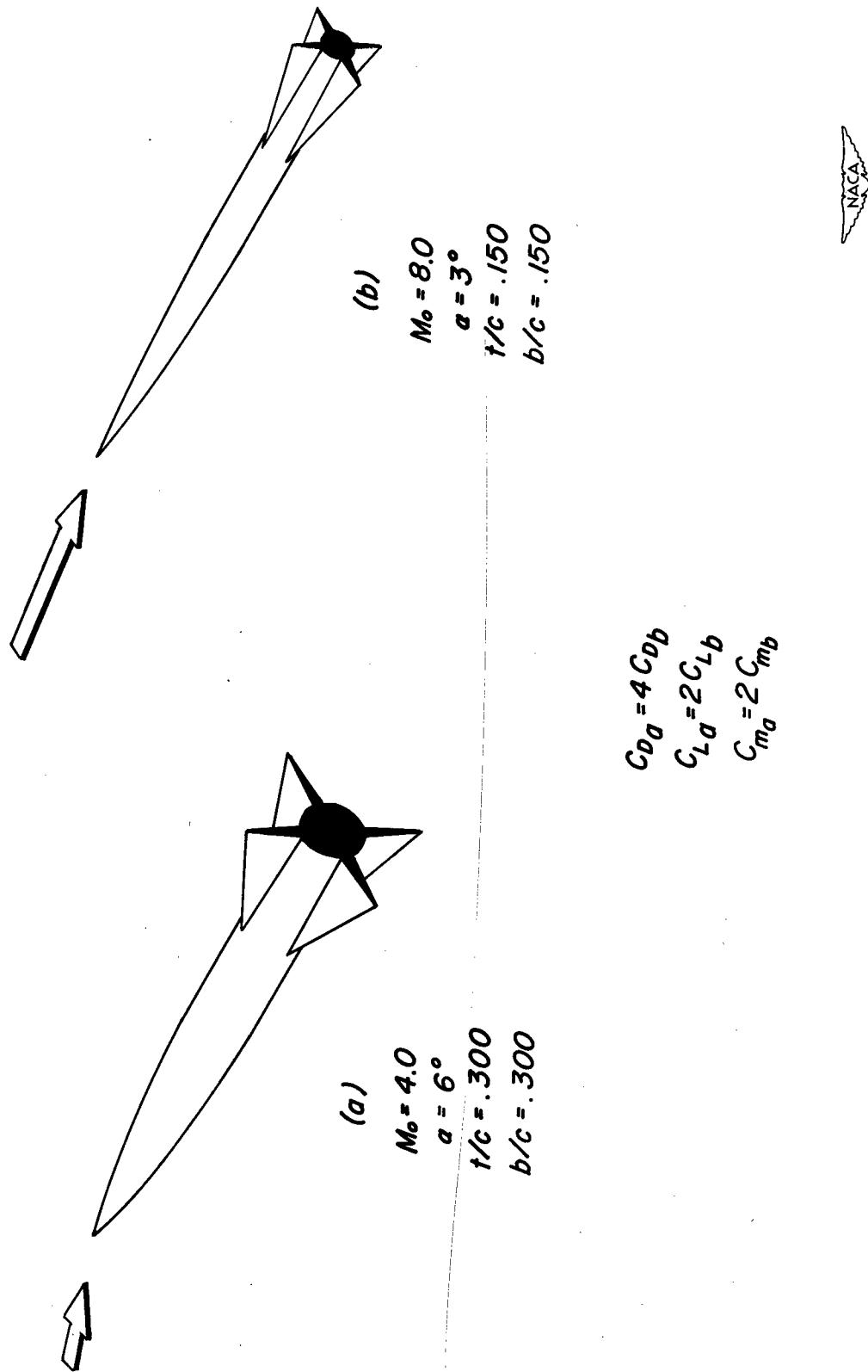


Figure 4.— Related wing-body combinations in hypersonic flow

NACA TN 2443
 National Advisory Committee for Aeronautics.
**THE SIMILARITY LAW FOR HYPERSONIC FLOW
 ABOUT SLENDER THREE-DIMENSIONAL SHAPES.**
 Frank M. Hamaker, Stanford E. Neice and A. J.
 Eggers, Jr. August 1951. 22p. diagrs. (NACA
 TN 2443)

The similarity law for steady hypersonic flow about slender three-dimensional shapes is derived in terms of customary aerodynamic parameters. The law is employed to determine simple relations for correlating the pressures, forces and moments acting on related shapes in hypersonic flows. In the special case of inclined bodies of revolution, these expressions for the forces and moments are generalized to include some significant effects of the viscous cross force.

Copies obtainable from NACA, Washington



NACA TN 2443
 National Advisory Committee for Aeronautics.
**THE SIMILARITY LAW FOR HYPERSONIC FLOW
 ABOUT SLENDER THREE-DIMENSIONAL SHAPES.**
 Frank M. Hamaker, Stanford E. Neice and A. J.
 Eggers, Jr. August 1951. 22p. diagrs. (NACA
 TN 2443)

1. Aerodynamics, Fundamental (1.1)
2. Flow, Supersonic (1.1.2.3)
3. Airplanes (1.7.1)
4. Missiles (1.7.2)
- I. Hamaker, Frank M.
- II. Neice, Stanford E.
- III. Eggers, A. J., Jr.
- IV. NACA TN 2443

The similarity law for steady hypersonic flow about slender three-dimensional shapes is derived in terms of customary aerodynamic parameters. The law is employed to determine simple relations for correlating the pressures, forces and moments acting on related shapes in hypersonic flows. In the special case of inclined bodies of revolution, these expressions for the forces and moments are generalized to include some significant effects of the viscous cross force.



NACA TN 2443

National Advisory Committee for Aeronautics.
**THE SIMILARITY LAW FOR HYPERSONIC FLOW
 ABOUT SLENDER THREE-DIMENSIONAL SHAPES.**
 Frank M. Hamaker, Stanford E. Neice and A. J.
 Eggers, Jr. August 1951. 22p. diagrs. (NACA
 TN 2443)

1. Aerodynamics, Fundamental (1.1)
2. Flow, Supersonic (1.1.2.3)
3. Airplanes (1.7.1)
4. Missiles (1.7.2)
- I. Hamaker, Frank M.
- II. Neice, Stanford E.
- III. Eggers, A. J., Jr.
- IV. NACA TN 2443

The similarity law for steady hypersonic flow about slender three-dimensional shapes is derived in terms of customary aerodynamic parameters. The law is employed to determine simple relations for correlating the pressures, forces and moments acting on related shapes in hypersonic flows. In the special case of inclined bodies of revolution, these expressions for the forces and moments are generalized to include some significant effects of the viscous cross force.

Copies obtainable from NACA, Washington



NACA TN 2443

National Advisory Committee for Aeronautics.
**THE SIMILARITY LAW FOR HYPERSONIC FLOW
 ABOUT SLENDER THREE-DIMENSIONAL SHAPES.**
 Frank M. Hamaker, Stanford E. Neice and A. J.
 Eggers, Jr. August 1951. 22p. diagrs. (NACA
 TN 2443)

1. Aerodynamics, Fundamental (1.1)
2. Flow, Supersonic (1.1.2.3)
3. Airplanes (1.7.1)
4. Missiles (1.7.2)
- I. Hamaker, Frank M.
- II. Neice, Stanford E.
- III. Eggers, A. J., Jr.
- IV. NACA TN 2443

The similarity law for steady hypersonic flow about slender three-dimensional shapes is derived in terms of customary aerodynamic parameters. The law is employed to determine simple relations for correlating the pressures, forces and moments acting on related shapes in hypersonic flows. In the special case of inclined bodies of revolution, these expressions for the forces and moments are generalized to include some significant effects of the viscous cross force.

Copies obtainable from NACA, Washington



NACA TN 2443
 National Advisory Committee for Aeronautics.
**THE SIMILARITY LAW FOR HYPERSONIC FLOW
 ABOUT SLENDER THREE-DIMENSIONAL SHAPES.**
 Frank M. Hamaker, Stanford E. Neice and A. J.
 Eggers, Jr. August 1951. 22p. diagrs. (NACA
 TN 2443)

1. Aerodynamics, Fundamental (1.1)
2. Flow, Supersonic (1.1.2.3)
3. Airplanes (1.7.1)
4. Missiles (1.7.2)
- I. Hamaker, Frank M.
- II. Neice, Stanford E.
- III. Eggers, A. J., Jr.
- IV. NACA TN 2443

The similarity law for steady hypersonic flow about slender three-dimensional shapes is derived in terms of customary aerodynamic parameters. The law is employed to determine simple relations for correlating the pressures, forces and moments acting on related shapes in hypersonic flows. In the special case of inclined bodies of revolution, these expressions for the forces and moments are generalized to include some significant effects of the viscous cross force.

Copies obtainable from NACA, Washington



NACA TN 2443
 National Advisory Committee for Aeronautics.
**THE SIMILARITY LAW FOR HYPERSONIC FLOW
 ABOUT SLENDER THREE-DIMENSIONAL SHAPES.**
 Frank M. Hamaker, Stanford E. Neice and A. J.
 Eggers, Jr. August 1951. 22p. diagrs. (NACA
 TN 2443)

1. Aerodynamics, Fundamental (1.1)
2. Flow, Supersonic (1.1.2.3)
3. Airplanes (1.7.1)
4. Missiles (1.7.2)
- I. Hamaker, Frank M.
- II. Neice, Stanford E.
- III. Eggers, A. J., Jr.
- IV. NACA TN 2443

The similarity law for steady hypersonic flow about slender three-dimensional shapes is derived in terms of customary aerodynamic parameters. The law is employed to determine simple relations for correlating the pressures, forces and moments acting on related shapes in hypersonic flows. In the special case of inclined bodies of revolution, these expressions for the forces and moments are generalized to include some significant effects of the viscous cross force.



Copies obtainable from NACA, Washington

